Lecture 4 - Complex Differentiation and Integration

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1 Force and Work

Consider a particle moving in a 2-dimensional force field

$$\mathbf{F} = (P, Q).$$

How much work do we need to do to move it along a path $\gamma(t) = (x(t), y(t))$? By definition, the amount of work done is given by the following line integral:

$$W = -\int_\gamma \mathbf{F} \cdot \mathbf{dr} = -\int_\gamma P dx + Q dy$$

Let's recall the mathematical definition of this integral.

Definition 1. *Let $D \subset \mathbb{R}^2$ be a domain. A 1-form on D is an expression*

$$P dx + Q dy$$

where P and Q are (complex valued) functions on D.

Given a 1-form $P dx + Q dy$ and a path $\gamma(t)$ in the region U, we define:

$$\int_\gamma P dx + Q dy = \int_a^b \left(P(\gamma(t)) \frac{dx}{dt} + Q(\gamma(t)) \frac{dy}{dt} \right) dt$$

Often in physics, the force field is given by negative gradient of a potential energy function:

$$\mathbf{F} = -(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y})$$

Such a force is usually called "conservative", for reasons we'll see in a minute.

Example 1. *A particle moving in a gravitational force field near the surface of the earth experiences a force:*

$$\mathbf{F} = (0, -mg)$$

where m is the mass of the particle and g is the Earth's gravitational constant. In this case, the potential energy is given by

$$U(x, y) = mgy$$

Example 2. *For a harmonic oscillator whose equilibrium point is located at the origin, the force field is*

$$\mathbf{F} = (-kx, -ky)$$

The potential energy for a harmonic oscillator is given by

$$U(x, y) = \frac{1}{2}kx^2 + \frac{1}{2}ky^2$$

In the case of a particle moving in a conservative force field, the work is given by

$$W = -\int_\gamma \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

There is a special notation for the 1-form appearing in the above integral.

Definition 2. *The* differential*, or* total derivative *of a function U is the 1-form*

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

Note that this definition is consistent with the use of the symbols dx and dy, because these are simply the differentials of the coordinate functions x and y. Some formal properties of differentials are collected below. They are left as exercises for the reader.

- $d(f+g) = df + dg$
- $d(fg) = fdg + gdf$
- The differential of a constant function is 0.
- $d(f^n) = nf^{n-1}df$

Note that differentials behave more or less exactly like derivatives. They are just a pleasant notation for talking about the partial derivatives of a function of several variables.

The reason why conservative forces are useful is that the work done by them can be computed using the Fundamental Theorem of Calculus. The idea is that the potential energy function behaves like an anti-derivative for the force field.

Theorem. *(Fundamental Theorem of Calculus for Line Integrals) Let U be a complex-valued function on a domain $D \subset \mathbb{R}^2$. Let $\gamma : [a,b] \to D$ be a path, and let $p = \gamma(a)$, $q = \gamma(b)$. Then*

$$\int_\gamma dU = U(q) - U(p)$$

Proof. By definition,

$$\int_\gamma dU = \int_a^b \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} dt$$

Let $f(t) = U(\gamma(t))$. Then by the chain rule, we have

$$\int_\gamma dU = \int_a^b \frac{df}{dt} dt$$

By the usual fundamental theorem of calculus,

$$\int_\gamma df = f(b) - f(a) = U(q) - U(p)$$

□

So, to figure out the work done by a conservative force field, one only needs to take the difference between the endpoints of the path:

$$W = -\int_\gamma \mathbf{F} \cdot \mathbf{dr} = \int_\gamma dU = U(q) - U(p)$$

Consider the case when $p = q$, i.e. when the path γ returns to its original position. In this case,

$$W = U(p) - U(p) = 0$$

So, no work is done as long as the particle returns to its original position. This is the reason for the term "conservative" - energy is conserved along paths that return to their original configuration. Such a system cannot have dissipative forces like friction, because such a force removes energy from the particle no matter which path is taken.

Paths which returning to their original positions have a special name in mathematics - they are called "closed contours" (or "closed paths").

2 Winding Numbers

Sometimes one wants to take differentials of things which are not quite functions, but are rather multivalued functions. For example, consider the "function" $\theta(x, y)$, which takes a vector (x, y) and returns the angle that it makes with the x-axis. We would like to say that

$$\theta(0,1) = \frac{\pi}{2}$$

$$\theta(-1,-1) = \frac{5\pi}{4}$$

But we might equally well say that

$$\theta(0,1) = \frac{5\pi}{2}$$

$$\theta(-1,-1) = \frac{-3\pi}{4}$$

In fact, any angle is only defined up to a multiple of 2π, so we should really say something like

$$\theta(0,1) = \frac{\pi}{2} + 2\pi k$$

$$\theta(-1,-1) = \frac{-3\pi}{4} + 2\pi k$$

where the number k is meant to be an undetermined integer. Hence $\theta(x, y)$ is properly thought of as a multi-valued function.

In fact it is impossible to define a continuous function $\theta(x, y)$. However, if such a function existed then we could compute its differential. Here's how we do it. First express x and y in polar coordinates and compute their differentials.

$$x = r\cos\theta \implies dx = \cos\theta dr - r\sin\theta d\theta$$

$$y = r\sin\theta \implies dy = \sin\theta dr + r\cos\theta d\theta$$

We can then solve for $d\theta$ as follows. First multiple the top and bottom equations by appropriate constants and subtract to remove the dr terms.

$$\cos\theta dy - \sin\theta dx = rd\theta$$

Then solve for $d\theta$ in terms of x and y and their differentials:

$$d\theta = \frac{r\cos\theta}{r^2}dy - \frac{r\sin\theta}{r^2}dx = \frac{xdy - ydx}{x^2 + y^2}$$

If you like, you can take this as the *definition* of $d\theta$ as a 1-form. Note that it is only defined on the punctured plane $A = \mathbb{R}^2 \setminus \{0\}$, because it has a singularity at the origin.

Now suppose we have a path $\gamma(t)$ in A. If γ is given by

$$\gamma(t) = r(t)e^{i\theta(t)}$$

for some smooth, real-valued functions $r(t)$ and $\theta(t)$, then we will have

$$\int_\gamma d\theta = \theta(t_1) - \theta(t_0)$$

So, integrating $d\theta$ measures the total angle passed through by the path $\gamma(t)$. As a consequence, if γ is a closed contour, then the integral must be an integral multiple of 2π!

Definition 3. *Let $A = \mathbb{R}^2 \setminus \{0\}$ and let γ be a closed contour in A. We define the* winding number *of γ around the point 0 as follows:*

$$w(\gamma, 0) = \frac{1}{2\pi} \int_\gamma d\theta$$

Obviously, an analogous definition applies if we puncture the plane at a point other than the origin, but then we must define $d\theta$ differently. In general, if $p = (x_0, y_0)$ is a point in the plane and γ is a closed contour not passing through $p = (x_0, y_0)$, then we can define

$$w(\gamma, p) = \frac{1}{2\pi} \int_\gamma \frac{(x - x_0)dy - (y - y_0)dx}{(x - x_0)^2 + (y - y_0)^2}$$

Intuitively, the winding number is the number of times that the contour wraps around the point p. This can usually be done by visually examining the contour, but it is comforting to know that it can also be done by evaluating an integral.

3 Visualizing $d\theta$

The best way to think about multi-valued functions is to think about their graphs. For example, the real variable function

$$f(x) = \sqrt{x}$$

takes two values for any positive value of x, namely the positive and negative square roots. Its "graph" therefore looks like a sideways parabola.

What is the graph of the function $\theta(x, y)$? It looks like a helical "parking garage" with infinitely many levels! (INSERT PICTURE)

How should we think about integrating $d\theta$?. In general, if we have a potential function U, its graph will be some surface lying over the $x - y$ plane. If we want to integrate dU along a path γ, we should "lift" the path γ to a path on this surface. As we move along this path, our height at time t is the value of the integral from t_0 to t.

So, when we're integrating $d\theta$ along a path γ, we should think of continuously lifting our path to an infinite parking garage instead of a single-sheeted surface. If we go around the origin once, then we find ourself "one level up" in the parking garage (or one level down, depending on whether we went clockwise or counter-clockwise). The winding number of a closed contour is the number of levels you go up as you traverse the contour.

4 Holomorphic Differentials and Integrals

Suppose that f is a holomorphic on a disk D. Then it has a power series expansion

$$f = \sum_{n=0}^{\infty} a_n z^n$$

Taking the differential, we get:

$$df = \sum_{n=0}^{\infty} a_n d(z^n) = (\sum_{n=0}^{\infty} n a_n z^{n-1}) dz$$

Note that the symbol dz is really just the differential of the function

$$z(x, y) = x + iy$$

In terms of dx and dy, it is given by:

$$dz = dx + idy$$

It makes sense to call the function in front of dz the "derivative" of the holomorphic function f. We write this as follows:

$$df = f'(z)dz = \frac{df}{dz}dz$$

This gives us a holomorphic function $f'(z)$ which is defined everywhere that f is. We can also define second, third, and higher order derivatives of holomorphic functions. Notice that any holomorphic function automatically has *infinitely many* derivatives.

An interesting fact is that holomorphic functions are characterized by the property that their differentials are proportional to dz.

Proposition 1. *Let $D \subset \mathbb{C}$ be a domain. A function $f : D \to \mathbb{C}$ is holomorphic if and only if there exists a function g such that $df = gdz$. In this case, the function g is holomorphic as well.*

Proof. It is convenient to introduce another differential form

$$d\overline{z} = dx - idy$$

It is then easy to check that

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) d\overline{z}$$

and by some linear algebra this expression is unique. Thus df is proportional to dz if and only if the second term vanishes, i.e. if and only if f satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

□

Anyway, differentiation of holomorphic functions is quite easy. You just write out a power series expansions and differentiate term-by-term. Anti-differentiation is also easy, provided the domain of definition for f is a disk. An anti-derivative for the function f above is

$$F = \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$$

Definition 4. *Let f be a holomorphic function defined on a domain $D \subset \mathbb{C}$. We say that f has a primitive (or anti-derivative) if there is another holomorphic function F such that*

$$dF = f(z)dz$$

Theorem. *Let f be a holomorphic function on a domain $D \subset \mathbb{C}$, and suppose that f has a primitive F on D. Then for any closed contour γ in D,*

$$\int_{\gamma} f(z)dz = 0$$

Corollary 1. *(Cauchy's Theorem, simple version) Let f be a holomorphic function on a disk D, and let γ be a closed contour in D. Then*

$$\int_{\gamma} f(z)dz = 0$$

5 The Residue Theorem

In general domains, not every holomorphic function has a primitive.

Lemma 1. *Let $A = \mathbb{C} \setminus 0$. Then the function $f = \frac{1}{z}$ is holomorphic on A but does not have a primitive on A.*

Proof. Suppose it did have a primitive F. We know that F has a Laurent series:

$$F = \sum_{n \in \mathbb{Z}} a_n z^n$$

Taking the differential of this, we get:

$$dF = \sum_{n<0} n a_n z^{n-1} + \sum_{n>0} n a_n z^{n-1}$$

There is no $\frac{1}{z}$ term anywhere in this series, so no matter how we choose the constants a_n we can't arrange that

$$dF = \frac{dz}{z}$$

Therefore, f does not have a primitive. □

Note that the above argument actually shows that a function

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

will have a primitive as long as $a_{-1} = 0$.

Corollary 2. *Let f be any holomorphic function on an annulus A centered at the origin. Then there is a holomorphic function F such that*

$$f(z)dz = dF + a_{-1}\frac{dz}{z}$$

The constant a_{-1} has a special name:

Definition 5. *Let f be a holomorphic function on an annulus centered at a point z_0. Then it has a Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

we define the residue *of f at z_0 to be the coefficient of $\frac{1}{z-z_0}$:*

$$\mathrm{Res}(f, z_0) = a_{-1}$$

There is actually a slicker way to see that $\frac{1}{z}$ can't have a primitive, which makes use of the fundamental theorem of calculus. Write the function z in polar coordinates as follows:

$$z = re^{i\theta}$$

Then compute its differential in terms of r and θ:

$$dz = e^{i\theta}dr + ire^{i\theta}d\theta$$

Dividing on the left by z and on the right by $re^{i\theta}$, we get:

$$\frac{dz}{z} = \frac{dr}{r} + id\theta = d(\log r) + id\theta$$

Now let γ be a closed contour that starts an ends at a point $r_0e^{i\theta_0}$. Then

$$\int_\gamma \frac{dr}{r} = \log r_0 - \log r_0 = 0$$

$$\int_\gamma id\theta = 2\pi i w(\gamma, 0)$$

Putting this together, we see that

$$\int_\gamma \frac{dz}{z} = 2\pi i w(\gamma, 0)$$

If $\frac{1}{z}$ had a primitive then its integral around any closed contour would be zero. Since the integral is not zero for paths with nonzero winding number, we are forced to conclude that $\frac{1}{z}$ does not have a primitive.

Theorem. *(Residue Theorem, simple version) Let f be a holomorphic function on an annulus centered at z_0, and let γ be a closed contour in this annulus. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \text{Res}(f, z_0)w(\gamma, z_0)$$

Proof. We know that there is a function F such that

$$f(z)dz = dF + \text{Res}(f, z_0)\frac{dz}{z}$$

Therefore,

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \frac{1}{2\pi i}\int_\gamma dF + \frac{Res(f, z_0)}{2\pi i}\int_\gamma \frac{dz}{z} = \text{Res}(f, z_0)w(\gamma, z_0)$$

□

In the case of a holomorphic function on a disk, we have seen that it has no negative terms in its laurent expansion, therefore its residue is zero, and we recover Cauchy's theorem as a special case.