Lecture 3 - The Heat, Wave, and Cauchy-Riemann Equations

Lucas Culler and Josh Batson

1 Differentiation and Convergence of Power Series

If one isn't concerned about convergence, it is easy to find the fourier expansion for the derivative of a function. Suppose that

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx}.$$

If we differentiate the sum term-by-term, we get

$$f'(x) = \sum_{n \in \mathbb{Z}} ina_n e^{inx}$$

So we see that the n-th fourier coefficient gets multiplied by a factor of in.

There is another way to see this, however, using the formula for the fourier coefficients:

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$$

Let b_n be the n-th fourier coefficient of $f'(x)$. Then

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} f'(x) e^{-inx} dx$$

Since the function f is periodic, there is no boundary term when we integrate by parts. Hence:

$$b_n = -\frac{1}{2\pi} \int_0^{2\pi} f(x) \left(-ine^{-inx} \right) dx = ina_n$$

The advantage of the second version is that it allows us to prove that smooth functions have convergent fourier series. Suppose that $f''(x)$ is a continuous function. Let a_n be the fourier coefficients of f and let c_n be the fourier coefficients of $f''(x)$. Then

$$c_n = -n^2 a_n$$

On the other hand, since $f''(x)$ is continuous, its absolute value is bounded above by some constant M. Therefore,

$$|c_n| = |\frac{1}{2\pi} \int_0^{2\pi} f''(x) e^{-inx} dx| \leq \frac{1}{2\pi} \int_0^{2\pi} M dx = M$$

So the numbers $|c_n|$ are bounded above by M as well. To prove that the fourier series converges, we are going to consider what happens if we cut the Fourier expansion off at some finite stage

$$\sum_{n=-N}^{N} a_n e^{in\theta}$$

How much does this finite sum differ from the infinite sum? Well, we have an upper bound on the absolute value of the "tail" or remainder of the series:

$$|\sum_{|n|\geq N} a_n e^{inx}| \leq \sum_{|n|\geq N} |a_n| = M \sum_{|n|\geq N} \frac{1}{n^2}$$

Now, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test. Hence its tails get arbitrarily small as N tends to infinity. But this implies that the tails of the fourier series become arbitrarily small as N tends to ∞. Therefore, the fourier series converges as well.

So, any sufficiently differentiable function will have a convergent fourier series. In fact, one can refine the arguments above to get even better convergence criteria, but since we are assuming all functions to be smooth we have no need for such analysis.

2 The Heat Equation

Suppose we have a metal ring, and we heat it up in some irregular manner, so that certain parts of it are hotter than others. Assume the ring is placed in some sort of insulating material, so that no heat is lost to the environment. How does the temperature at different points in the ring change over time? To answer this question, suppose that the temperature at time t and at an angle θ relative to a fixed point on the ring is given by a function $u(t, \theta)$. Then the time evolution of u can be modelled by a partial differential equation, called the "heat equation":

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial \theta^2}$$

where κ is a constant depending on the size and physical properties of the ring. As it turns out, this equation is what motivated Fourier to invent Fourier series, and indeed they are still the best way to solve it. His method goes like this. First, expand u as a Fourier series with time-dependent coefficients:

$$u(\theta, t) = \sum_{n\in\mathbb{Z}} A_n(t) e^{in\theta}$$

Note that the coefficients A_n can be complex, even though $u(\theta, t)$ is assumed to be real. Next, substitute this expression into the heat equation:

$$\sum_{n\in\mathbb{Z}} \frac{dA_n}{dt} e^{in\theta} = \sum_{n\in\mathbb{Z}} A_n \kappa (in)^2 e^{in\theta}$$

Since Fourier series are unique, we can set these two expressions equal term by term, obtaining (infinitely many!) ordinary differential equations for the coefficients A_n:

$$\frac{dA_n}{dt} = -\kappa n^2 A_n$$

Solving these equations gives us:

$$A_n(t) = c_n e^{-\kappa n^2 t}$$

for some constants c_n. Substituting back in, we get a formula for u:

$$u(\theta, t) = \sum_{n\in\mathbb{Z}} c_n e^{-\kappa n^2 t + in\theta}$$

So we are left to determine the constants c_n. How can we do this? Well, setting $t = 0$, we get

$$u(\theta, 0) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

So if we know the initial temperature distribution then we can find the constants c_n by taking its Fourier transform:

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} u(\theta, 0) e^{-in\theta} d\theta$$

Thus Fourier series provide us with a thorough, systematic way to solve the heat equation with periodic boundary conditions.

What does this solution tell us physically? Taking the limit as $t \to \infty$ shows that the temperature of the ring approachs a constant, namely c_0. Moreover, looking at the constants in the exponentials we see that the higher Fourier modes die off much quicker than the lower modes. So rapid oscillations get smoothed out rather quickly, whereas more modest oscillations will take a longer time to reach equilibrium.

One peculiar feature of the heat equation is that it cannot be solved infinitely far back in time. The reason for this is that the Fourier series describing the solution ceases to be convergent when the coefficients grow too large. However, our argument shows that a solution does exist for all positive times, and this solution is uniquely determined by the initial conditions.

3 The Wave Equation

Now suppose we take our metal ring and strike it with a hammer. The ring will then vibrate, meaning that it will be displaced from its original shape, and the displacement will change with time, oscillating back and forth, presumably at a fairly high frequency. The displacement $\phi(\theta, t)$ of the ring can be modelled (assuming dissipative effects are negligible) by the following partial differential equation, called the "wave equation":

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial \theta^2}$$

where c is some constant depending on things like the density of the ring and its radius. Let's again use Fourier series to solve this equation. Since the displacement is periodic, we can write it as follows:

$$\phi(\theta, t) = \sum_{n \in \mathbb{Z}} A_n(t) e^{in\theta}$$

where A_n are complex valued functions. We can then plug this into the equation above, obtaining:

$$\sum_{n \in \mathbb{Z}} \frac{d^2 A_n}{dt^2} e^{in\theta} = \sum_{n \in \mathbb{Z}} A_n(t) c^2 (in)^2 e^{in\theta}$$

Setting coefficients equal to one another we obtain:

$$\frac{d^2 A_n}{dt^2} = -c^2 n^2 A_n$$

In this case, the ordinary differential equation has two independent solutions:

$$A_n(t) = a_n e^{inct} + b_n e^{-inct}$$

so that the full solution is given by:

$$\phi(\theta, t) = \sum_{n \in \mathbb{Z}} a_n e^{in(\theta + ct)} + \sum_{n \in \mathbb{Z}} b_n e^{in(\theta - ct)}$$

So we see that ϕ can be written as a sum of two arbitrary functions

$$\phi(\theta, t) = f(\theta + ct) + g(\theta - ct)$$

which correspond physically to waves of displacement propogating in the clockwise and counterclockwise directions.

Even though we can eliminate Fourier series from our answer at the end, the Fourier decomposition of the wave is still interesting from a physical point of view. In particular, note that the function

$$\phi(\theta, t) = e^{in(\theta+ct)}$$

returns to its original configuration whenever ct is a multiple of $\frac{2\pi}{n}$. So, the sound waves emitted by the ring will have frequencies

$$\nu_n = \frac{nc}{2\pi}$$

Thus there will be a lowest or fundamental frequency, with overtones occuring at integer multiples of that frequency. Of course, we should really have allowed the ring to have a circumference L other than 2π, in which case the frequencies would be

$$\nu_n = \frac{nc}{L}$$

This shows that if we make the ring wider, then its fundamental frequency decreases. In other words, the note we hear when we strike it has a lower pitch. Of course, this phenomenon is familiar from our experiences with musical instruments, and so on.

4 The Cauchy-Riemann Equations

Now that we have seen how to use Fourier series to solve partial differential equations of physical importance, let's try to apply the same method to the case we really care about, the Cauchy-Riemann equation. Recall that this equation is:

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

First let's try to find all solutions that are periodic in the x variable. We proceed exactly as before. First we write f as a Fourier series in x:

$$f(x, y) = \sum_{n\in\mathbb{Z}} A_n(y)e^{inx}$$

Then we use the Cauchy-Riemann equations to get ordinary differential equations satisfied by the functions $A_n(y)$:

$$inA_n = -i\frac{dA_n}{dy}$$

Solving these, we get:

$$A_n = c_n e^{-ny}$$

and therefore f is given by

$$f(x, y) = \sum_{n\in\mathbb{Z}} c_n e^{-ny+inx} = \sum_{n\in\mathbb{Z}} c_n e^{in(x+iy)}$$

This can be expressed in a form that makes it evidently holomorphic:

$$f(z) = \sum_{n\in\mathbb{Z}} c_n e^{inz}$$

So, periodic holomorphic functions have Fourier series in much the same way as periodic real-valued functions have Fourier series.

Note that the Fourier series of a periodic holomorphic function will only converge when y is constrained to lie in some interval, because the positive Fourier coefficients increase exponentially with y and the negative Fourier coefficients decrease exponentially with y. We will exploit this momentarily when we prove the Riemann extension theorem and the Liouville theorem.

Theorem. *Let f be a holomorphic function defined on an annulus $A = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$. Then f is given on A by a Laurent series (that is, a power series with both positive and negative powers):*

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n$$

Moreover, the coefficients c_n are uniquely determined by the function f.

Proof. The idea is to use "logarithmic coordinates". More concretely, consider the infinite horizontal strip

$$S = (-\infty, \infty) \times (\log r_1, \log r_2)$$

There is a conformal map $g : S \to A$ given by $z \mapsto e^{iz}$. Given any holomorphic function f on A, we therefore get a periodic holomorphic function $f \circ g$ on S. By the computation above, $f \circ g$ can be written as follows:

$$f(g(z)) = \sum_{n \in \mathbb{Z}} c_n e^{inz} = \sum_{n \in \mathbb{Z}} c_n g(z)^n$$

for some constants c_n. Setting $w = g(z)$, we get:

$$f(w) = \sum_{n \in \mathbb{Z}} c_n w^n$$

for all $w \in A$, as claimed. □

We often want to know about holomorphic functions on a disk rather than an annulus. The following result says that in this case the laurent series expansion has only nonnegative powers of z.

Theorem. *(Riemann Extension Theorem) Let f be a holomorphic function defined on a punctured disk $D^* = \{z \in \mathbb{C} : 0 < |z| < r\}$. Suppose moreover that f is bounded by some constant, i.e. $|f(z)| < M$ for all $z \in D^*$. Then f is given by a power series*

$$f(z) = \sum_{n \geq 0} c_n z^n$$

In particular, we can use the power series expansion to extend f to a holomorphic function defined on the entire disk D, and this extension is unique.

Proof. The preimage of D^* under e^{iz} is an upper half-space $H = \{z = x + iy : y > -\log r\}$. Using the standard formula for Fourier coefficients, we get:

$$c_n e^{-ny} = \frac{1}{2\pi} \int_0^{2\pi} f(x + iy) e^{-inx} dx$$

So if f is bounded by a constant M, then we get:

$$|c_n e^{-ny}| \leq M$$

for all $y > \log r$. Hence if n is negative, c_n must be zero, which shows that the Laurent expansion of f only has nonnegative terms. □

Corollary 1. *Any holomorphic function f defined on an open disk D has a power series expansion valid on that disk:*

$$f(z) = \sum_{n \geq 0} c_n z^n$$

Corollary 2. *(Liouville's Theorem) Any bounded, holomorphic function defined on the entire complex plane is constant.*

Proof. Proceeding as in the proof of the Riemann extension theorem, we see that for every $n \in \mathbb{Z}$ we must have

$$|c_n e^{-ny}| \leq M$$

for some constant M. Since the expression on the left gets arbitrarily large when $y \to \pm\infty$, we must have $c_n = 0$ for all nonzero n. Therefore, f is constant. □