LECTURE 2

1. Outline

Fourier decompositions of periodic functions.

- Why is the Fourier decomposition possible?
  - cardinality: how many functions are there $I = [-\pi, \pi] \to \mathbb{R}$? ($\aleph_2$, a real numbers worth of real numbers) Continuous? ($\aleph_1$ a natural numbers worth of real numbers) Differentiable?
  - How can we represent functions as lists of numbers, in such a way that simple/important/common functions have simple/short representations?
  - Record values at points, linear interpolate/spline.
    - simple functions require as much data as terribly complex ones.
  - Polynomial approx
    - but simple reps of polynomials aren't the simplest functions, especially if we have a periodic input. $\sin x$ should be simple, not x^2.
  - Trigonometric approx
    - $\sum b_n \cos nx + c_n \sin nx$
  - View the space of functions as a vector space (adding coordinate-wise in linear rep, polynomial rep, trigonometric rep)
    - looking for a basis.
- Why is the Fouier decomposition plausible?
  - sinusoids form an algebra, so we can get products
    - use double angle formulas, any other ideas you have to express $\sin^2 x = (1 - \cos 2x)/2$, $\sin 2x \cos x$ as sums of $\sin nx$ and $\cos nx$.
    - to prove this, helps to $\frac{1}{2}(e^{inx} + e^{-inx}) = \cos x$ and $\frac{1}{2i}(e^{inx} - e^{-inx}) = \sin x$. Allow ourself to have complex coefficients.
  - shifted sines: $\sin(x + \pi/3) = \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$
  - Can get a Dirac delta-ish function, with a single spike in one place. ideas?
    - first try to center it at π. $\sin^{2N} x$ has two (positive) peaks, add $\sin^{2N+1} x$ to keep just one.
    - or $\sqrt{\frac{n}{4\pi}}\left(\frac{1+\cos x}{2}\right)^n$ to make it positive first.

- Why does the Fourier decomposition work?
  - okay, so we can get all the products of sines and cosines, shifted versions, etc. but why can we approximate/decompose *all* periodic functions?
  - BASIS VECTORS:
$$f_n = \frac{1}{\sqrt{2\pi}} e^{inx}$$
  - INNER PRODUCT: need a formula to compute coefficients. in finite dim geometry, use dot product to get coords: e_1-coord is $v \bullet e_1$. What is the appropriate notion of dot product for functions? Called the L^2-norm. For $f, g : I \to \mathbb{R}$, define
$$\langle f, g \rangle := \int_{-\pi}^{\pi} fg dx$$
    Why is this good? Well, when you dot a function with itself, you should get its magnitude, which I'll call the ENERGY of f:
$$\|f\|^2 := \langle f, f \rangle = \int_{-\pi}^{\pi} |f|^2 dx$$
    The only function with 0 energy is the 0 function. Triangle inequality from the same for regular absolute value. Also, dot product should be linear in each input, which this integral is.
    * For $f, g : I \to \mathbb{C}$, add complex conjugation:
$$\langle f, g \rangle := \int_{-\pi}^{\pi} f\bar{g} dx.$$
  - Our basis vectors are ORTHONORMAL:
$$\langle f_n, f_m \rangle = \frac{1}{2\pi} \int e^{inx} e^{-imx} dx = \frac{1}{2\pi} \int e^{i(n-m)x} dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$
    * If we were right and $f = \sum a_n f_n$, then $\langle f, f_m \rangle = \sum a_n \langle f_n, f_m \rangle = a_m$. So we can *find the fourier components of f via integration*:
$$a_m = \langle f, f_m \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f e^{-imx} dx.$$
    * *Because of Dirac delta approx*, if f is continuous, periodic, and orthogonal to the span of the (f_n), then $f = 0$!
    * PROTOTHEOREM: If $\sum a_n f_n$ is a function, then $f - \sum a_n f_n$ is too, and is orthogonal to thes pan of the (f_n) so zero.
    * But the sum might not converge, if we're handed a function f and compute all the a_n, is it the case that $f = \frac{1}{\sqrt{2\pi}} \sum a_n e^{inx}$? We don't know anything about the a_i. We know something:
$$0 \le \left\langle f - \sum a_n f_n, f - \sum a_n f_n \right\rangle = |f|^2 - 2 \sum |a_n|^2 + \sum |a_n|^2$$
      Well, do the partial sums approximate f? Leaving out something about Cauchy, we can compute the norm of
$$f - \sum_{n=-N}^{N} a_n f_n = |f|^2 - \sum_{n=-N}^{N} |a_n|^2.$$

- Why is the Fourier decomposition useful?
  - interaction with differentiation. If $f = \sum a_n e^{inx}$, then $f'(x) = \sum ina_n e^{inx}$; $a_n \rightsquigarrow ina_n$, no limits necessary!
  - C^2 gives pointwise convergence.
    - * Bessel for f'' gives $a_n < 1/n^2$ gives uniform convergence to a function, and we can use above.

Theorem. *If $f : I \to \mathbb{C}$ is a smooth, periodic function, then $f = \sum a_n e^{inx}$ for some constants $a_n \in \mathbb{C}$, and*

$$\int_{\pi}^{\pi} |f|^2 \, dx = \sum |a_n|^2 .$$

Fourier

- hilbert space description: periodic functions form an infinite-dimensional vector space. the set of square-integrable ones, though, have a countable basis (!) given by
  - question: how many functions are there on S^1? How about if we ask them to be continuous? Differentiable? Integrable? Square integrable?
- Daniel Bernulli noticed, about 1750, in his study of the acoustic problem of vibrating strings, that the general vibration of a string could be represetned by the superposition of those sine vibrations which corresponded to the fundamental tone and the overtones. The development into a trigonometric series of the function which represents the form of the string.
- fourier had the balls to believe
- sines, cosines, and sums provide examples of periodic functions
  - even $\sin^2 x = (1 - \cos 2x)/2$
- fourier decomposition: any function can be approximated arbitrarily well by a sum of sines and cosines
  - $f = \sum a_n \sin(nx) + b_n \cos(nx)$ only positive n since cos is even, sin is odd
  - but sines and cosines are real and imaginary parts of complex numbers, we can write
  - $$\cos x = \frac{e^{ix} + e^{-ix}}{2} \text{ and } \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
  - They form an algebra!
- interaction with differentiation
  - nice to have an orthonormal basis, so you can use coordinates and inner products will be coordinate-wise dot products

Proposition 1. *The set $B = \{\sin nx, \cos nx\}_{n=0,1,2,...}$ forms a complete basis for the Hilbert space $L^2[-\pi, \pi]$.*

Proof. First, we construct a test function in the span of B which has a sharp spike at one point (more than 2/3 of its mass in an arbitrarily small window). Consider

$$g_n(x) = \sqrt{\frac{n}{4\pi}} \left(\frac{1 + \cos x}{2} \right)^n .$$

Note that g_n is everywhere positive, with a peak at 0. For n large, $\int g_n \simeq 1$. And

$$\int_{-1/\sqrt{n}}^{1/\sqrt{n}} g_n \to_{n\to\infty} 0.8427\ldots$$

To prove this, use the half-angle formula

$$\cos x = \cos 2(x/2) = 2\cos^2 x/2 - 1$$

to rewrite

$$g_n(x) = \sqrt{\frac{n}{4\pi}}\cos^{2n} x/2.$$

Now $\int_{-\pi}^{\pi} \cos^{2n} x/2 = 2\int_{-\pi/2}^{\pi/2} \cos^{2n} x$ and we can use complex exponentials to rewrite $\cos^{2n} x$:

$$\begin{aligned}
\cos^{2n} x &= (\frac{e^{ix}+e^{-ix}}{2})^{2n} \\
&= \frac{1}{2^{2n}}\sum_{m=0}^{2n}\binom{2n}{m}e^{(2n-2m)ix} \\
&= \frac{1}{2^{2n}}\sum_{k=-n}^{n}\binom{2n}{n+k}\cos 2kx
\end{aligned}$$

Rescaling, we find that $g_n(x)$ is a sum of terms of the form

$$\frac{\binom{2n}{n+k}}{2^{2n}}\sqrt{\frac{n}{4\pi}}\cos 2kx$$

For fixed k, and large n, Stirling's formula tells us that the terms approach

$$\frac{1}{2\pi}\cos 2kx.$$

Regardless, the integral of $\cos 2kx$ from $-\pi/2$ to $\pi/2$ vanishes for all integers k, so we are left with only the constant term, which integrates out to 1.

$$\int_{-\pi}^{\pi} g_n(x) \sim 1.$$

To find out how much of the weight is concentrated near the origin, it helps to use a power series expansion.

$$\log\cos x = -\frac{x^2}{2} + x^4 r(x)$$

for some smooth function r, so

$$\begin{aligned}
\int_{-2/\sqrt{n}}^{2/\sqrt{n}} g_n(x)dx &= \sqrt{\frac{n}{4\pi}}\int_{-2/\sqrt{n}}^{2/\sqrt{n}} \cos^{2n}(\frac{x}{2})dx \\
&= \frac{1}{\sqrt{\pi}}\int_{-1}^{1} \cos^{2n}(\frac{y}{\sqrt{n}})dy \quad \text{where } y = \sqrt{n}x/2 \\
&= \frac{1}{\sqrt{\pi}}\int_{-1}^{1} \exp\left[-y^2 + y^4/n^2 r(y/\sqrt{n})\right] dy \\
&\sim \frac{1}{\sqrt{\pi}}\int_{-1}^{1} \exp\left[-y^2\right] dy \\
&\sim 0.8427\ldots
\end{aligned}$$

□

This is secretly the same argument used to prove the central limit theorem!!