COVERING MAPS

1. Examples of Covering Maps

- Last time, we showed that any holomorphic map f can be written as $f(z) = f(z_0) + w(z - z_0)^n$ in some neighborhood around z_0. After a conformal change of coordinates in the domain and range, this looks like $f(z) = z^n$. The points z_0 for which $n > 1$ are called **ramification points**. Their images are called **branch points**. If we remove branch points from the range and their preimages from the domain, then we have a conformal map of surfaces.
- These maps $f : X \mapsto Y$ have an especially nice property: if we look at any small region (say, a disk) in the range, its preimages are a bunch of disjoint disks in the domain.
- Definition:$\pi : X \mapsto Y$ is a covering map if every point $p \in Y$ is contained in some region U whose preimage is a disjoint collection of identical copies of $U_1,, U_2, \ldots$ of U, and π maps each copy bijectively onto U.
- Examples
  - Helix
  - Finite $S^1 \to S^1$
  - Trivial covering spaces
  - Exp
  - $z \mapsto z^n$ on punctured disk, sphere
  - $z \mapsto z^3 - 3z$
  - Corresponding cover of the figure 8
  - Other covers of the figure 8 (for the audience)
  - $z \mapsto z^4 - z^3$ gives bouquet of 5 circles onto bouquet of 2
  - Genus 3 covering genus 2
  - Genus 4 covering genus 2
  - $\mathbb{R}^2$ over torus
- Nonexample
  - $z \mapsto z^n$ on the whole disk
  - circle squashed onto segment

2. The Disk has no Nontrivial Covers

2.1. Subdivision.

- Covers of the interval are trivial. If both halves are trivial, glue to get trivial. If not, smaller segment. Eventually converge to a point, but it has a trivializing neighborhood.
- Cor: we've found all the covers to the circle.
- Covers of the square are trivial. If both halves are trivial, glue to get trivial. If not, smaller segment. Eventually converge to a point, but it has a trivializing neighborhood.

- Covers of the sphere are trivial. Cut into two disks. Each has trivial cover, and disks get glued to make

2.2. **Path Lifting.**

- Walking around the cover looks like walking around the base: any small patch you're standing on is identical to the area around its image.
- **Unique path lifting property**: a conga line wending its way in the base has a unique conga line on top.
- A space is **simply connected** if any circle of people standing in it, holding hands, can all walk to a fixed destination without breaking the chain.
  - The disk and sphere are simply connected–walk to the centre/south pole in a straight line.
- Simply connected spaces have only trivial covers. Chooseγ in X. Since X is sc, the conga line can collapse without breaking. run that backwards to have them expand out from the first guy without breaking. Their paths all lift uniquely, so get some curve $\tilde{\gamma}'$ upstairs, which projects to γ and, since last guy started holding hands with the first guy, $\tilde{\gamma}'$ is closed upstairs, and the two bps are the same)

Next lecture, use branch cuts to make your surface less branch points simply connected, and see where those "fundamental domains" are lifted to.

3. Highbrow

Two ways to understand spaces:

- Geometry: the study of spaces with structure
  - Metric structure (distances)
  - Conformal structure (angles)
- Topology: the study of unstructured (squishy) spaces

Klein's Erlangen program said that to understand a space, we should study the invertible self-maps which preserve its structure

- Euclidean space (with its metric structure)–>Group of rotations and translations
- Sphere (with its metric structure)–>Group of rotations (of $\mathbb{R}^3$) ($PSU(2) = SO(3)$)
- Sphere (with its conformal structure)–>Mobius transformations $z \mapsto \frac{az+b}{cz+d}$($PSL(2, \mathbb{C})$)
- Complex plane (with its conformal structure)–>Group of rotations, translations, and dilations (since infinity fixed, and well-behaved near it: 1-1 means that infinity gets mapped to infinity, so $1/f(1/z)$ is holo and bounded in a disk near infinity, so by riemann extension the map extends to the riemann sphere)
- Hyperbolic plane (with its metric structure)–>$PSL(2, \mathbb{R})$
- Hyperbolic plane (with its conformal structure)–>$PSL(2, \mathbb{R})$

Modern mathematics says that to really understand a space, we should study the structure-preserving maps into and out of it (not just $1-1$, not just to self)

- Holomorphic maps into and out of a space.
  - Holomorphic maps are locally $c + z^n$ after a conformal change of coordinates

    - Away from branch points, map is conformal, so 1-1, sheets stretch and cover segments
- The number of covering spaces is determined by *topology* of the space, not the geometry! and given a topological covering, there is a unique geometry induced by a choice of geometry on the base.
    - conformal coverings of the punctured disk/annulus given by coverings of the circle. once you pick the conformal structure on the annulus, you get the conformal structure on the cover.