# Lecture 7 - The Riemann Sphere

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## 1 Spherical Coordinates

Until now, we have considered holomorphic and conformal mappings of domains in the complex plane. We are now going to take a bold move outside of this context, and step into the third dimension. Consider the unit sphere $S^2 \subset \mathbb{R}^3$. (PICTURE)

In rectangular coordinates, the sphere is defined by the following equation:

$$x^2 + y^2 + h^2 = 1$$

Since we are reserving the letter $z$ for complex numbers, we use $h$ for the height above the $x$-$y$ plane.

Given a point $p = (x, y, h)$ on the sphere, define $r(p)$ to be the distance from the point $p$ to the vertical line that passes through the north and south poles of the sphere. Algebraically,

$$r = x^2 + y^2$$

The number $r(p)$ is called the polar radius.

Now define $\theta(p)$ to be the angle that the vector $(x, y)$ makes with the $x$-axis. In other words,

$$x + iy = re^{i\theta}$$

The angle $\theta$ is called the polar angle. Together, the height $h$ and the polar angle $\theta$ constitute "cylindrical coordinates" on the sphere.

Another quantity of interest is the angle $\phi(p)$ made by the $x$-$y$ plane and the line $L_p$ that passes through the center of the sphere and the point $p$. This angle is restricted to lie in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and is given by the formula:

$$e^{i\phi} = r + ih$$

The angle $\phi(p)$ is called the azimuthal angle. When some people define it they shift by a phase of $\frac{\pi}{2}$, so be careful if you are used to a different convention. Together, the polar angle $\theta$ and the azimuthal angle $\phi$ constitute "spherical coordinates" on the sphere.

## 2 Conformal Mapping on $S^2$

If we have two vectors tangent which are tangent to $S^2$ at the same point, then we can measure the angle between them, just by drawing the vectors in $\mathbb{R}^3$ and measuring the angle there. So, given a domain $D \subset S^2$ we can say what it means for a map $f : D \to \mathbb{C}$ to be angle-preserving.

Recall, however, that for a map between planar domains to be conformal it had to preserve angles *as measured in the counterclockwise direction.* On the sphere, there is an issue about which direction is clockwise and which is counterclockwise. The ambiguity arises from whether we decide to look at the sphere from the inside or from the outside. What looks like a counterclockwise rotation from the outside looks like a clockwise rotation from the inside.

Ah, but there's the lie! It turns out we had this same issue in the plane, but chose to ignore it. We have implicitly been making a decision to look at the complex plane from the top rather than from the bottom. If we had chosen to look at it from the bottom, then all our counterclockwise measurements would become clockwise measurements.

So, to resolve our difficulties on the sphere, we just need to make an arbitrary choice about whether to look at it from inside or outside. The most natural convention is to look at it from the outside, and this is what we'll do.

Now let $D \subset S^2$ be a domain not containing the north or south poles. Then the polar and azimuthal angles define coordinates on $D$. We seek a differential equation like the Cauchy-Riemann equations that will tell us when a map $f : D \to \mathbb{C}$ is holomorphic. To do this, define $e_\theta$ and $e_\phi$ to be the unit tangent vectors in the polar and azimuthal directions, respectively. Where are these vectors taken by the mapping $f$?

First let's do the azimuthal direction. Consider the path $\gamma(t) = (\theta, t)$, where $\theta$ is fixed and $t \in (-\pi, \pi)$, and we have written $\gamma$ in spherical coordinates $(\theta, \phi)$. The velocity of this path is $e_\phi$, since it has speed 1 and always points in the azimuthal direction. Under the mapping $f$, the vector $e_\phi$ should be taken to the velocity of the path $f(\gamma(t))$ at $t = \phi$. So, under the mapping $f$,

$$\mathbf{e}_\phi \mapsto \frac{d}{dt}|_{t=\phi} f(\theta, t) = \frac{\partial f}{\partial \phi}(\theta, \phi)$$

Next let's do the polar direction. Consider the path $\gamma(t) = (\frac{t}{r}, \phi)$ where $\phi$ is fixed and $t$ lies in $[0, 2\pi)$. This path traverses a circle of radius $r$ at an angular velocity of $\frac{1}{r}$, so it travels at unit speed as well. Since it always points in the polar direction, its velocity is always given by $e_\theta$. Therefore, under the mapping $f$,

$$\mathbf{e}_\theta \mapsto \frac{d}{dt}|_{t=\theta} f(\frac{t}{r}, \phi) = \frac{1}{r}\frac{\partial f}{\partial \theta}(\theta, \phi)$$

Note that $e_\phi$ is obtained from $e_\theta$ by doing a counterclockwise rotation of 90 degrees. Therefore, $\frac{\partial f}{\partial \phi}$ should be obtained from $\frac{1}{r}\frac{\partial f}{\partial \theta}$ by doing a counterclockwise rotation of 90 degrees. Therefore,

$$\frac{\partial f}{\partial \phi} = \frac{i}{r}\frac{\partial f}{\partial \theta}$$

These are the Cauchy-Riemann equations on a sphere! Let's try to solve them using Fourier series. We can write $f$ as a Fourier series in $\theta$:

$$f(\theta, \phi) = \sum_{n \in \mathbb{Z}} c_n(\phi) e^{in\theta}$$

If we apply the Cauchy-Riemann equations to this equation (exercise!), we get the following infinite sequence of ODE:

$$\frac{\partial c_n}{\partial \phi} = \frac{-nc_n}{r} = \frac{nc_n}{\cos \phi}$$

Sadly, this is not a differential equation that we immediately know how to solve. We will have to attack it cleverly.

## 3 Solving the CR equations on $S^2$

We need to solve for the coefficient functions $c_n(\phi)$. First let's do the case $n = 1$. Let $R(\phi) = c_1(\phi)$ be a function such that $R(0) = 1$ and such that $R$ satisfies the differential equation:

$$\frac{\partial R}{\partial \phi} = \frac{R}{\cos \phi}$$

Then we have

$$\frac{dR}{R} = \frac{d\phi}{\cos\phi}$$

Let's do the change of variables $z = e^{i\phi}$ and use Euler's formula, rather than muck around with trig functions. Under this change of variables,

$$\frac{dz}{z} = id\phi$$

and

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2} = \frac{z + z^{-1}}{2}$$

so the differential equation reads

$$\frac{dR}{R} = 2i\frac{1}{z + z^{-1}}\frac{dz}{z} = 2i\frac{dz}{z^2 + 1}$$

At this point, we're tempted to use the trig function $\tan^{-1}(z)$, but let's continue to think about complex numbers. Instead, do a partial fraction decomposition of $\frac{1}{z^2+1}$:

$$\frac{1}{z^2 + 1} = \frac{1}{2i}\frac{1}{z - i} - \frac{1}{z + i}$$

This gives us:

$$\frac{dR}{R} = \frac{dz}{z - i} - \frac{dz}{z + i}$$

Therefore,

$$\log R = \log(z - i) - \log(z + i) + \log C$$

for some nonzero constant C. exponentiating both sides, we get:

$$R = C\frac{z - i}{z + i}$$

and substituting $z = e^{i\phi}$ gives us:

$$R = C\frac{e^{i\phi} - i}{e^{-i\phi} + i}$$

a lot of trigonometry finally gives

$$R = C\frac{i\cos\phi}{1 + \sin\phi}$$

Choosing $C = -i$ now gives the final solution:

$$R(\phi) = \frac{cos\phi}{1 + \sin\phi} = \frac{r}{1 + h}$$

whew! And that was just the first equation. Now we have to solve for $c_n$. Of course, we could go through the whole thing again, but a better idea is to change variables from $\phi$ to $R$. When we do this, we get:

$$\frac{dR}{R} = \frac{d\phi}{\cos\phi}$$

and therefore,

$$\frac{dc_n}{c_n} = n\frac{dR}{R}$$

Integrating both sides gives us:

$$\log(c_n) = n\log(R) + C_n$$

Exponentiating, we get:

$$c_n(\phi) = C_n R^n$$

So, the general solution of the Cauchy-Riemann equations is:

$$f(\theta, \phi) = \sum_{n \in \mathbb{Z}} c_n R^n e^{in\theta}$$

It makes sense to define $Z(\theta, \phi) = Re^{i\theta}$. Then the general solution takes the form of a power series in $Z$:

$$f = \sum_{n \in \mathbb{Z}} c_n Z^n$$

Thus we can express any holomorphic function $f$ on $S^2$ as a power series in the function $Z$.

# 4 Stereographic Projection

The function $Z$ emerged mysteriously from our computations, but we can also derive it from a geometric construction known as stereographic projection. Given a point $p$ on $S^2$, we draw a straight line connecting it with the south pole. This line intersects the complex plane in a unique point $Z(p)$.

(PICTURE)

To check that this is the same as the mapping $Z$ defined earlier, first note that a circle moving around the $z$-axis in a counterclockwise manner gets sent to a circle in the complex plane moving around the origin in a counterclockwise manner. Therefore,

$$Z(re^{i\theta}, h) = R(p)e^{i\theta}$$

for some function $R(p)$. To determine this function, we use the similar triangles as indicated in the picture below.

(PICTURE)

This tells us that

$$R = \frac{R}{1} = \frac{r}{1+h}$$

and therefore,

$$Z = \frac{re^{i\theta}}{1+h}$$

which is the same as the $Z$ we derived from abstract principles. In particular this shows that stereographic projection is a conformal mapping.

# 5 Rational Functions

Using the mapping $Z$, we can freely identify the complex plane with $S^2 \setminus \{\text{southpole}\}$. When we think about the sphere in this way, we will call it the Riemann sphere and denote it by $\widehat{\mathbb{C}}$. From this perspective, the complex plane is really a punctured sphere - a sphere with a hole poked in it at the south pole.

**Definition 1.** *Let $D \subset \widehat{\mathbb{C}}$ be a domain. A "meromorphic function" on $D$ is a continuous function $f : D \to \widehat{C}$ which is holomorphic and takes finite values away from a discrete set of points $S \subset D$. The set $S$ is called the singular locus of $f$.*

Note that there is a local condition for being meromorphic, namely that for every point $p \in D$ the function is holomorphic on a punctured disk containing $p$ and has a well-defined (but possibly infinite) limit as $z \to p$.

How can we check that a function is meromorphic on the Riemann sphere? Well, checking it on $\mathbb{C}$ is easy; we just use our Laurent series expansions and check that they all have finite tails. The only difficulty is checking it at infinity. But for this case we can use the conformal change of coordinates $w = \frac{1}{z}$ and check that the function is meromorphic at $w = 0$ in the usual way.

Another way to phrase the above discussion is to say that a function $f : D \to \widehat{C}$ is meromorphic if and only if both of the functions $f(z)$ and $f(\frac{1}{z})$ are meromorphic away from $z = \infty$.

**Example 1.** *Any "rational function"*

$$f(z) = \frac{p(z)}{q(z)}$$

*where $p$ and $q$ are polynomials, is a meromorphic function on $S^2$. This can be seen by doing a "partial fractions" decomposition of the numerator, for example.*

**Example 2.** *Sums, products, and quotients of meromorphic functions are meromorphic (as long as you don't divide by the zero function!).*

**Definition 2.** *Let $f$ be a meromorphic function on the punctured disk. Then there is some integer ord$(f)$,*

$$f(z) = z^{\mathrm{ord}(f)} h(z)$$

*where $h$ is holomorphic and $h(0) = 0$. This integer is called the* order *of $f$.*

If $f$ has a pole, then it has a negative order. If it has a zero, then it has a positive order.

**Lemma 1.** *Let $f : \widehat{C} \to \mathbb{C}$ be holomorphic. Then $f$ is constant.*

*Proof.* First I claim that any continuous complex-valued function on a sphere is bounded. To prove this, cut up the sphere into finitely many triangles, each of diameter less than $\frac{1}{10}$. If the function were bounded on each triangle then it would be bounded on the sphere. Hence it must be unbounded on some triangle. Cut up this triangle into finitely many smaller triangles, this time with diameter less than $\frac{1}{100}$. Again the function must be unbounded on one of the tiny triangles. Finding smaller and smaller triangles on which the function is unbounded, we can zoom in on a point where the function blows up (and hence the function can't be continuous at that point).

So, any holomorphic function $f : \widehat{C} \to \mathbb{C}$ must be bounded. But then it is constant by Liouville's theorem. $\square$

**Theorem.** *Let $f : \widehat{C} \to \widehat{C}$ be meromorphic. Then $f$ is a rational function.*

*Proof.* For every point $p \in \mathbb{C}$ let $n_p$ be the order of $f$ at $p$. Then consider the function:

$$F(z) = f(z) \cdot \prod_{p \in \mathbb{C}} (z - p)^{-n_p}$$

The zeroes and poles of $f$ in the complex plane are all cancelled out by the new factors, so $F(z)$ is holomorphic and nonzero on the entire complex plane. But it is also meromorphic, hence it has a limit as $z \to \infty$.

Case 1: If $F$ is holomorphic at infinity, then it is holomorphic on the entire Riemann sphere, hence it must be constant. Therefore,

$$f(z) = M \prod_{p \in \mathbb{C}} (z - p)^{n_p}$$

for some constant $M$ as desired.

Case 2: If $F(\infty) = \infty$, then $\frac{1}{F}$ is holomorphic at infinity. Since $F$ is holomorphic and nonzero everywhere in the complex plane, $\frac{1}{F}$ is holomorphic everywhere in the complex plane as well. Hence $\frac{1}{F}$ is constant as well, and we proceed as in case 1 to conclude that $f(z)$ is rational. $\square$

# 6 Applications

As an application of our theory, we can prove the famous fundamental theorem of algebra:

**Theorem.** *Let $f(z)$ be a polynomial with complex coefficients. Then $f$ factors completely into linear factors:*

$$f(z) = (z - \alpha_1) \cdot (z - \alpha_2) \cdots (z - \alpha_n)$$

*Proof.* Observe that any polynomial is a meromorphic function on the Riemann sphere, since it tends to $\infty$ as $z \to \infty$. The proof of the above theorem then implies that

$$f(z) = \prod_{\alpha \in \mathbb{C}} (z - \alpha)^{n_\alpha}$$

and since $f$ is holomorphic on $\mathbb{C}$ it has no poles, so all $n_\alpha$ are positive. Thus $f$ factors into linear factors as claimed. $\square$

As another application, we can classify all conformal bijections $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

**Theorem.** *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a conformal bijection. Then there exist constants $a, b, c, d$ with $ad - bc \neq 0$ such that*

$$f(z) = \frac{az + b}{cz + d}$$

*Proof.* If a rational function is bijective, then it can only have one zero, so its numerator must be a linear polynomial. Likewise it can only have one pole, so its denominator must be nonzero as well. So

$$f(z) = \frac{az + b}{cz + d}$$

If $ad - bc = 0$ then by linear algebra $az + b$ is a scalar multiple of $cz + d$, so $f$ is constant, hence not bijective. Thus $ad - bc \neq 0$ as claimed. $\square$

We finish with an application to cartography. The Mercator projection is often used in map-making to create...well, to create conformal maps. It is a conformal mapping $f : \mathbb{C} \to S^2$ satisfying

$$f(z + 1) = f(z)$$

$$f(z) \to \text{n.p. as } \mathrm{Im} z \to \infty$$

$$f(z) \to \text{s.p. as } \mathrm{Im} z \to -\infty$$

$$f(\mathrm{Im} z = 0) = \text{equator}$$

$$f(\mathrm{Re} z = 0) = \text{date line}$$

In fact, it is given by

$$f(z) = Z^{-1}(e^{2\pi i z})$$

where $Z^{-1}$ is the inverse of the stereographic projection map.

**Theorem.** *The Mercator projection is unique.*

*Proof.* As always, we use $Z$ to identify $S^2$ with $\widehat{C}$. Suppose that both $f$ and $g$ satisfies the properties of the Mercator projection. Then $F = f \circ g^{-1}$ would be a conformal bijection from the Riemann sphere to itself. Hence it must be given by

$$F(z) = \frac{az + b}{cz + d}$$

for some $a, b, c, d$. But it takes zero to zero and $\infty$ to $\infty$. Therefore, $b = 0$ and $c = 0$. Hence

$$F(z) = re^{i\theta} z$$

for some scaling factor $r$ and some rotation angle $\theta$. Hence

$$f(z) = re^{i\theta} g(z)$$

as claimed. $\square$