THE LAPLACE TRANSFORM

1. Fourier Transform

Recall the Fourier transform: given a periodic signal $f : [-\pi, \pi] \to \mathbb{C}$, we can split it into component frequencies

$$f = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} a_n e^{inx},$$

where

$$a_n = \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Unfortunately, most signals we encounter in real life do not have perfect periodicity, even a "pure" pitch is really a mixture of nearby frequencies. We usually have some signal $f : \mathbb{R} \to \mathbb{R}$, and could compute any of it's frequency components:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx.$$

This new function $\hat{f}$ is the Fouier transform of f. If f eventually dies off (in particular if $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$), then these integrals are finite, and we can write our original signal as an *integral* of its pure frequency components:

$$f(x) = \frac{1}{2\pi} \int \hat{f}(\xi) e^{i\xi x} d\xi.$$

You can think of this integral as a limit of discrete fourier transforms, on longer and longer intervals, allowing for slower and slower oscillations (smaller frequencies). Just like a bandlimited signal is determined by its samples, a compactly supported function is determined by sufficiently fine samples of its fourier transform.

1.1. **An aside.**

What is the fourier transform of a pure exponential function, e^{inx}? The integral never converges, but its conventional to say that the answer is a delta distribution $\delta(n - \xi)$. What about a pure exponential which is truncated after a few cycles, say $e^{inx}\chi_{[0,2\pi k]}$. An explicit computation gives

$$\hat{f} = i \frac{1 - e^{-2\pi i k \xi}}{n - \xi}$$

Away from $\xi = n$, this is just the wave $1 - e^{2\pi i k \xi}$ scaled down by $\frac{1}{n-\xi}$. A zero of the wave is placed on top of the pole, so near $n = \xi$, we are computing the derivative of the numerator at $n = \xi$, which gives $2\pi k$. Somehow, this acts like a delta function, as $k \to \infty$, though it never has a narrow envelope.

We would like to use fourier series to develop the fourier transform, prove its invertibility. If we truncate a function to $[-\pi k, \pi k]$ and divide the integral by k, then we recover the original fourier transform for functions with period 2π at the samples $\xi \in k\mathbb{Z}$. Looking at the samples $\xi \in \mathbb{Z}$ allows us to see functions with period $2\pi k$. Any compactly supported

2. Continue, Analytically

If we've learned anything in this class, it's that expanding our domain from the real numbers to the complex numbers can clarify things. And here, with the Fourier transform, we're taking our function f and integrating it against a purely imaginary exponential: $e^{i\xi x}$ to get a new function of ξ. Let's think of this as a function defined on the imaginary axis: we take in the imaginary number $i\xi$and compute $\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$. But we could just as easily take in any complex number z and compute

$$F(z) = \int_{-\infty}^{\infty} f(x)e^{-zx}dx.$$

Since e^{-zx} is a holomorphic function of z, and the integral is essentially a sum (with weights given by f), it turns out that $F(z)$ is holomorphic! If you remember Lucas's lecture on Fourier transform, where we learned that knowing the value of a holomorphic function on a single strip can be enough to determine it, we're explicitly constructing the unique holomorphic extension of the Fourier transform.

This function F is the Laplace transform. But before we write it as a definition, let's change notation a bit: since we're thinking of f as a signal, as a function of time, we usually write t instead of x,and the convention is also to write s for z (since it makes people who are only thinking of real values for s a bit more comfortable). To wit:

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The **Laplace Transform** of $f(t)$ is the function of a complex variable

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt.$$

Our shiny new extension of the Fourier transform has some great properties:

- The map $f \rightsquigarrow F$ is linear: $\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g]$
- It is (basically) injective $\mathcal{L}[f] = \mathcal{L}[g]$ then $f = g$ almost everywhere.
- The Laplace transform F is holomorphic everywhere it's defined.

But that domain of definition can be a bit narrow. Consider the constant function $f = 1$. The integral

$$F(z) = \int_{-\infty}^{\infty} e^{-zx}dx$$

is defined for no values of z! None! For if $z = a + bi$, we're trying to compute

$$\int_{-\infty}^{\infty} e^{-ax}e^{-ibx}dx.$$

If $a > 0$, then the magnitude of the integra

nd gets huge as $x \to -\infty$, and if $a < 0$, then the magnitude of the integrand gets huge as $x \to \infty$. If $a = 0$, then the magnitude is always 1, and the integral still fails to converge. So we've found a nowhere defined holomorphic function.

Luckily, most signals we get in the real world don't extend infinitely into the past (or the future, for that matter), we only begin to measure them at some time $t = t_0$. The prototypical example is the unit step function

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

The value at $t = 0$ can be left undefined, or chosen arbitrarily. This represents a switch turning on, or a phenomenon tuned in ("oh, let's start looking at that steady signal.") Think of it as a smooth function whose value changes from 0 to 1 in a faster time scale than we're observing, like the velocity of a ball struck off a tee with a bat (we ignore the strange things happening in a slight switch before a steady current begins to flow.) Its Laplace transform is

$$\begin{aligned} U(s) &= \int_{-\infty}^{\infty} u(t)e^{-st}dt \\ &= \int_0^{\infty} e^{-st}dt \\ &= \left.\frac{e^{-st}}{-s}\right|_0^{\infty} \\ &= \frac{1}{s} \text{ if } \Re(s) > 0 \end{aligned}$$

If $\Re(s) \le 0$, then the integral does not converge. Miraculously, though, the function we get for $s > 0$ can be extended to an analytic function on the entire complex plane, less the origin. Compare to the Fourier transform, which cannot be defined: we get nonconvergent integrals

$$\int_0^{\infty} e^{-i\xi t}dt.$$

In other words, the infinity we get should be thought of as $\int_0^\infty 1dt = \lim_{s\to 0}\frac{1}{s}$.

The basic functions for which we want to compute the Laplace transform are then not functions like e^{at} or $\sin(x)$ or t^n which are defined on the whole real line, but versions of those functions which get switched on at some time, which we take to be 0. Here we go

Example 1. Exponential shift. We compute $\mathcal{L}(e^{at})$ by answering a more general question: if $\mathcal{L}[f] = F(s)$, what is $\mathcal{L}(e^{at}f)$?

$$\begin{aligned} \mathcal{L}[e^{at}f] &= \int e^{at}fe^{-st}dt \\ &= \int fe^{-(s-a)t}dt \\ &= F(s-a) \end{aligned}$$

In particular, $\mathcal{L}[e^{at}] = \frac{1}{s-a}$.

Example 2. Sinusoids. We can write $\cos(\omega t)$ as $\frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$, so

$$\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) \rightsquigarrow \frac{1}{2}(\frac{1}{s-i\omega} + \frac{1}{s+iw}) = \frac{s}{s^2+\omega^2}.$$

Similarly,

$$\sin(\omega t) \rightsquigarrow \frac{\omega}{s^2+\omega^2}.$$

We saw that the Fourier transform interacted very nicely with differential equations, and derivatives in general. So does the more general Laplace transform, for the same reason: integration by parts.

$$\begin{aligned}\mathcal{L}[f'] &= \int f'(t)e^{-st}dt \\ &= fe^{-st}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(-se^{-st})dt \\ &= s\mathcal{L}[f]\end{aligned}$$

as long as f vanishes for early enough times. We need to be a little bit careful when dealing with discontuities (which we basically always have, due to the switch flipping at 0). In particular, the derivative of the unit step function u, though it vanishes everywhere by at 0, is definitely not 0. Indeed, the fundamental theorem of calculus tells us that

$$\int_{-\epsilon}^{\epsilon} u'(t) = u(\epsilon) - u(-\epsilon) = 1.$$

We call the derivative of u the unit step impulse function or Dirac delta function, and write it $\delta(t)$. If we think of u as a smooth function going very quickly from 0 to 1, then its derivative δ zooms from 0 just before time 0 to some huge value just around 0, then back down to 0 just after. It looks like a very narrow tall spike centered at 0, with total integral one. We can find the Laplace transform of δ from the rule for derivatives we just computed:

$$\mathcal{L}[\delta] = s\mathcal{L}[u] = s/s = 1.$$

3. Inverting the Laplace Transform

If f has a fourier transform, then its Laplace transform is defined on the imaginary axis, and we may compute

$$\frac{1}{2\pi i}\int_{s=-i\infty}^{+i\infty} F(s)e^{st}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\xi)e^{i\xi t}d\xi = f(t).$$

If f does not have a fourier transform, if it doesn't decay fast enough, then damp it, consider $f_\sigma = fe^{-\sigma t}$ for some large σ. As long as f does not grow super-exponentially, then our f_σ has a fourier transform $\hat{f}_\sigma$, and the fourier inversion formula states that

$$f_\sigma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}_\sigma(\xi)e^{i\xi t}d\xi$$

Now the Laplace transform of $f_\sigma = fe^{-\sigma t}$ is $F(s+\sigma)$ where F is the Laplace transform of f, and $F_\sigma(i\xi) = \hat{f}_\sigma(\xi)$. So we have

$$f(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\xi+\sigma)e^{i\xi t}d\xi.$$

Change variables to $s = i\xi + \sigma$, and bring the $e^{\sigma t}$ under the integral to find

$$f(t) = \frac{1}{2\pi i}\int_{s=\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds.$$

Miraculously the answer is the same no matter which σ we used, so long as f_σ has an fourier transform. That's because F is defined and holomorphic for such $s = \sigma + i\xi$, and so is Fe^{st}. Hence the contour we integrate along can be shifted left and right through this region of holomorphicity without affecting the value of the integral.

4. Differential Equations

The unit impulse function is very useful in studying differential equations. For example, consider a spring with a mass on it, a damped harmonic oscillator, governed by the differential equation

$$mx''(t) + cx'(t) + kx(t) = 0.$$

It's sitting at rest for $t < 0$, and then at time 0, we whack it with a hammer, imparting a impulse of 1, so that it has unit momentum at time ϵ. Then its momentum $p(t) = mx'(t)$ jumps from 0 to 1, and can be modelled as a unit step function. Thus $p'(t) = mx''(t)$ is the derivatve of the unit step function, ie the unit impulse $\delta(t)$. So to understand how the spring moves in response to our thwack, we need to solve the differential equation

$$mx''(t) + cx'(t) + kx(t) = \delta(t).$$

Take the Laplace transform of each side to get

$$ms^2 X(s) + csX(s) + kX(s) = 1,$$

or

$$X(s) = \frac{1}{ms^2 + cs + k}.$$

Suppose we can factor the polynomial $P(s) = ms^2 + cs + k$ as $m(s - a)(s - b)$. Then we can rewrite

$$X(s) = \frac{1}{m(b-a)} \left(\frac{1}{s-a} - \frac{1}{s-b} \right).$$

But we know which function has that Laplace transform. It's just a sum of exponentials:

$$x(t) = \frac{1}{m(b-a)} \left(e^{as} - e^{bs} \right) u.$$

For example, if there is no damping ($c = 0$), then $a, b = \pm i\sqrt{k/m}$, and

$$x(t) = \frac{1}{2\sqrt{km}} \sin(\sqrt{k/m}t)$$

for $t > 0$ and 0 for $t < 0$. If $c > 0$ and the discriminant of P, $c^2 - 4mk$ is negative, then there is damped oscillatory behavior, while if the discriminant is positive, $c^2 > 4mk$, then x is a sum of purely real, and decaying, exponentials, and there is no oscillation at all.

The Laplace transform is helpful in understanding more general linear differential equations as well. It turns

$$P(D)f = \sum a_n f^{(n)}(t) = g$$

into a polynomial equation

$$P(s)F(s) = G(s)$$

where F and G are the Laplace transforms of f and g, and $P(s)$ is the polynomial $\sum a_n s^n$. To solve the differential equation, all we need to do is compute the inverse Laplace transform of $F(s) = \frac{G(s)}{P(s)}$. When g is the unit impulse function, $G = 1$, and we're trying to find a function w such that

$$W(s) = \frac{1}{P(s)}.$$

The function W is called the **transfer function** for the differential equation, and it's inverse Laplace transform $w(t)$ is called the **fundamental solution**.

How do we find the fundamental solution? Using partial fraction decomposition. A pole in the transfer function to the right of the imaginary axis indicates an instability in the system, at that frequency.

References

[Mattuck] http://ocw.mit.edu/OcwWeb/Mathematics/18-03Spring-2006/VideoLectures/detail/embed19.htm