# DISCRETE FT IMPLIES CONTINUOUS FT

Let $f : [-\pi, \pi] \to \mathbb{C}$ be a periodic function. If $f \in C^2$ then

$$f = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} a_n e^{inx},$$

where

$$a_n = \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Now let $f : \mathbb{R} \to \mathbb{C}$ be smooth with compact support, say, supp$f \subset [-K\pi, K\pi]$ for some large integer $K$. Then $g(x) = f(Kx)$ is supported on $[-\pi, \pi]$ and is periodic on the same. Thus

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{inx} dx$$

where

$$\begin{aligned}
\hat{g}(n) &= \int_{-\pi}^{\pi} g(x) e^{-inx} dx \\
&= \int_{-\pi}^{\pi} f(Kx) e^{-inx} dx \\
&= \int_{-K\pi}^{K\pi} f(y) e^{-iny/K} \frac{dy}{K} \text{ where } y = Kx \\
&= \frac{1}{K} \hat{f}(\frac{n}{K})
\end{aligned}$$

Replacing $g$ with $f$ in the previous formula yields

$$f(x) = g(\frac{x}{K}) = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{inx/K} = \sum_{n=-\infty}^{\infty} \hat{f}(\frac{n}{K}) e^{inx/K} \frac{1}{K}$$

The right hand side, when $K \to \infty$, is a Riemann sum for the integral

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi$$

with rectangles drawn atop intervals of length $\frac{1}{K}$ . After a certain $K_0$, every $K$ satisfies supp$f \subset [-K\pi, K\pi]$, and the limit is of identical terms all with value $f(x)$. Since smooth functions with compact support are dense in $L^2$(the square-integrable functions), and the fourier transform is an isometry on the smooth functions, it extends to an isometry on all of $L^2$. Hence for all $L^2$ functions $f$,

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi.$$